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THE RESIDUAL NILPOTENCE OF VERBAL WREATH PRODUCTS

BY

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Dedicated to the memory of Professor Brian Hartley

ABSTRACT

We obtain necessary and sufficient conditions for residual nilpotence of verbal wreath products. We study also residual nilpotence of the groups of the form $F/V(N)$, where F is a free group, $N \triangleleft F$, and $V(N)$ is a verbal subgroup of N.

1. Introduction

Let G be an arbitrary group, H be a group, free in some variety of groups $\mathfrak{U}$. We study in this paper residual properties of verbal wreath product $F = H_{\mathrm{wr}_{\mathfrak{U}}} G$ (see the definition of the verbal wreath products in H. Neumann [15] or in Section 2 of this paper). Our main results are necessary and sufficient conditions for the residual nilpotence of verbal wreath products, i.e. conditions when $F \in \mathrm{res}\mathfrak{N}$, where $\mathfrak{N}$ is the class of nilpotent groups (Theorem C), and necessary and sufficient conditions for $F \in \mathrm{res}\mathfrak{N}_p$, where $\mathfrak{N}_p$ is the class of nilpotent p-groups of the bounded exponent (Theorem A).

To formulate Theorem B we have to recall first the concept of a generalized periodic element (see A. I. Malcev [14]). A group S is residually nilpotent if

$$\bigcap_{k=1}^{\infty} \gamma_k(S) = 1$$

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(here $\gamma_k(S)$ denotes as usual the kth term of the lower central series of S); an element $s \in S$ is called generalized periodic (or periodic, for short) if it has a finite order in every quotient group $S/\gamma_k(S)$, i.e. if $f^{m_k} \in \gamma_k(S)$ for an appropriate number m_k; if all the numbers m_k are powers of p, then f is a p-element. If S is a residually nilpotent group without generalized periodic elements, then S is residually torsion free nilpotent.

An element $s \in S$ has an infinite p-height in S if the equation

$$g^{p^m} = s$$

has a solution for all $m \geq 1$ in each quotient group $S/\gamma_k(S)$. The set of all elements of infinite p-heights is a subgroup of S which coincides with the subgroup $\bigcap_{m=1}^{\infty} \bigcap_{n=1}^{\infty} S^{p^m} \gamma_n(S)$.

We will formulate now Theorem B. We observe first that if $F = H_{\mathrm{wr}_{\mathfrak{U}}} G$ and $F \in \mathrm{res}\mathfrak{N}$ then we see immediately that $G \in \mathrm{res}\mathfrak{N}$ and $H \in \mathrm{res}\mathfrak{N}$. Hence in the study of the residual nilpotence of the verbal wreath products $F = G_{\mathrm{wr}_{\mathfrak{U}}} H$ we must assume that G and H are residually nilpotent. Furthermore, if G and H are residually torsion free nilpotent, then so is $F = H_{\mathrm{wr}_{\mathfrak{U}}} G$ by Šmelkin's Theorem 6 in [18]. Hence, we can consider in fact only the case when one of the groups G or H contains a periodic element, say f. Let $m_k(f)$ be the orders of f in the quotient groups $G/\gamma_k(G)$ or $H/\gamma_k(H)$ respectively ($k = 1, 2, \ldots$). Let $p_i(f)$ ($i \in I$) be all the prime divisors of all the numbers $m_k(f)$.

THEOREM B: *Let $F = H_{\mathrm{wr}_{\mathfrak{U}}} G$. Then F is residually nilpotent iff one of the following conditions holds:*

(i) *Both groups G and H are residually torsion free nilpotent.*

(ii) *The group G contains an element g of finite order. Then $F \in \mathrm{res}\mathfrak{N}$ iff there exists a prime p such that $G \in \mathrm{res}\mathfrak{N}_p$, $H \in \mathrm{res}\mathfrak{N}_p$ (and hence the order of g must be in fact a power of p as well as the finite orders in G and H).*

(iii) *H contains elements of finite order. If all these orders are powers of the same prime number p, then $F \in \mathrm{res}\mathfrak{N}$ iff $G \in \mathrm{res}\mathfrak{N}_p$ and $H \in \mathrm{res}\mathfrak{N}_p$; if H contains elements with relatively prime orders, then $F \in \mathrm{res}\mathfrak{N}$ iff H is abelian of finite exponent, say $n = p_1^{m_1} p_2^{m_2} \cdots p_l^{m_l}$ ($l \geq 2$) and for every $i = 1, 2, \ldots, l$, $G \in \mathrm{res}\mathfrak{N}_{p_i}$.*

(iv) *Both groups G and H are torsion free and there exists a generalized periodic element f in one of these groups. Then $F \in \mathrm{res}\mathfrak{N}$ iff for every given non-unit elements*

$$g_\alpha \in G, \quad h_\beta \in H \quad (\alpha = 1, 2, \ldots, r; \beta = 1, 2, \ldots, s) \tag{1.1}$$

there exists a prime number $p_i = p_i(f)$ such that no one of the elements (1.1) has an infinite p_i-height in G and H respectively.

If the group H is residually torsion free nilpotent (and relatively free), then it is easy to see that the element of it cannot have an infinite p-height for any prime number p. We obtain that in this case the condition (iv) of Theorem B can be reformulated as the following one. For every given non-zero element

$$g_\alpha \quad (\alpha = 1, 2, \ldots, r)$$

there exists a prime number $p_i = p_i(f)$ such that no one of the elements (1.1) has an infinite p_i-height in G. This is equivalent via [11] to the residual nilpotence of the augmentation ideal $\varpi(ZG)$. We obtain from this the following corollary of Theorem B.

COROLLARY 1: *Assume that the group H is residually torsion free nilpotent. Then $F \in \operatorname{res}\mathfrak{N}$ iff $\bigcap_{k=1}^{\infty} \varpi^k(ZG) = 0$.*

Now let S be a free group, $N \lhd S$ be a normal subgroup, $G = S/N$, $V(N)$ be a verbal subgroup of N. The residual nilpotence of the groups of the type $\bar{S} = S/V(N)$ was studied in a number of papers. It began with Gruenberg's work [4] for the case when $\bar{S} = S/N'$ and $(S\colon N)$ is finite; in this case $\bar{S} \in \operatorname{res}\mathfrak{N}$ iff $(S\colon N)$ is a power of p. Lichtman proved in [11] that for an arbitrary normal subgroup $N \lhd S$ the group $\bar{S} = S/N'$ is residually nilpotent iff $\bigcap_{k=1}^{\infty} \omega^k(ZG) = 0$, where $G = S/N$. The necessary and sufficient conditions for the residual nilpotent of the augmentation ideal $\omega(ZG)$ were also obtained in [11]: the group G either must be residually torsion free nilpotent or $G \in \operatorname{dis} G_i$ $(i \in I)$, where G_i is a nilpotent p_i-group of a bounded exponent. Here the notation $G \in \operatorname{dis} G_i$ $(i \in I)$ means that G is discriminated by the system of groups G_i $(i \in I)$, i.e. for every given non-unit element $g_1, g_2, \ldots, g_r$ a group G_i and a homomorphism $\varphi_i\colon G \longrightarrow G_i$ can be found such that

$$\varphi_i(g_\alpha) \neq 1 \quad (\alpha = 1, 2, \ldots, r).$$

C. K. Gupta and N. D. Gupta [5] and Hartley [7] considered the cases $\bar{S} = S/\gamma_k(N)$ and $S = \bar{S}/V(N)$, where $V(N) \supseteq \gamma_k(N)$ and $N/V(N)$ is torsion free. They proved that also in these cases $\bar{S} \in \operatorname{res}\mathfrak{N}$ iff $\bigcap_{k=1}^{\infty} \varpi^k(ZG) = 0$.

We obtain from Theorem B sufficient conditions for the residual nilpotence of the groups $\bar{S} = S/V(N)$, where $V(N)$ is an arbitrary verbal subgroup of N. In fact these conditions follow immediately from Theorem B together with

Šmelkin's imbedding theorem, which states that $S/V(N)$ is imbedded into the verbal wreath product $H_{\mathrm{wr}_{\mathfrak{U}}}G$, where $H = N/V(N)$ and $\mathfrak{U}$ is the variety of groups generated by H.

We have hence the following corollary of Theorem B.

COROLLARY 2: *Let S be a free group, $N \lhd S$ be its normal subgroup, $G = S/N$. Let $V(N)$ be a fully invariant subgroup of N, $\bar{S} = S/V(N)$, $H = N/V(N)$. Then the group $\bar{S}$ is residually nilpotent provided that one of the conditions* (i)–(iv) *holds. In particular, if H is residually torsion free nilpotent and $\bigcap_{k=1}^{\infty} \varpi^k(ZG) = 0$, then $\bigcap_{n=1}^{\infty} \gamma_n(\bar{S}) = 1$.*

We conjecture that all the conditions of Theorem B, except the commutativity in case (ii), are necessary for the residual nilpotence of $\bar{S}$.

Now let once again $F = H_{\mathrm{wr}_{\mathfrak{U}}}G$. We obtain also necessary and sufficient conditions for the residual nilpotence of the ideal $\varpi(ZF)$.

THEOREM C: *The ideal $\varpi(ZF)$ is residually nilpotent iff one of the following three conditions holds:*

(i) *Both groups G and H are residually torsion free nilpotent.*
(ii) *There exists an element f of finite order in one of the groups G or H. Then the order of f must be a power of a prime number p and $G \in \mathrm{res}\mathfrak{N}_p$, $H \in \mathrm{res}\mathfrak{N}_p$.*
(iii) *The groups G and H are torsion free and there exists a generalized periodic element f in one of these groups. Then $\bigcap_{k=1}^{\infty} \varpi^k(ZF) = 0$ iff for every given non-unit element* (1.1) *there exists a prime number $p_i = p_i(f)$ such that no one of these elements has an infinite p_i-height.*

Comparing Theorem C with necessary and sufficient conditions for the residual nilpotence of the free product $G * H$, obtained by Lichtman in [12], we obtain immediately

COROLLARY 1: *The following conditions are equivalent:*

(i) $\bigcap_{k=1}^{\infty} \varpi^k(ZF) = 0$.
(ii) $\bigcap_{k=1}^{\infty} \varpi^k Z(G \times H) = 0$.
(iii) *The free product of groups $G * H$ is residually nilpotent.*

The results from [12] now yield also the second corollary of Theorem C.

COROLLARY 2: *Assume that the element f has a finite order or, more generally, the set $p_i(f)$ $(i \in I)$ is finite. Then the ideal $\varpi(ZF)$ is residually nilpotent iff there exists among the numbers p_i $(i \in I)$ a number p such that $G \in \mathrm{res}\mathfrak{N}_p$, $H \in \mathrm{res}\mathfrak{N}_p$.*

The same argument as in Corollary 1 of Theorem B yields

COROLLARY 3: *Assume that the group H is residually torsion free nilpotent. Then $\bigcap_{n=1}^{\infty} \varpi^n(ZF) = 0$ iff $\bigcap_{n=1}^{\infty} \varpi^n(ZG) = 0$.*

The crucial step in the proofs of Theorems B and C is Theorem A.

THEOREM A: *Let $F = H_{\mathrm{wr}_{\mathfrak{U}}} G$. Then $F \in \mathrm{res}\mathfrak{N}_p$ iff $G \in \mathrm{res}\mathfrak{N}_p$ and $H \in \mathrm{res}\mathfrak{N}_p$.*

Once again, we obtain immediately the following corollary:

COROLLARY OF THEOREM A: *Let S be a free group, $N \lhd S$, $G = S/N$, $V(N)$ be a verbal subgroup of N, $\bar{S} = S/V(N)$. Then $\bar{S} \in \mathrm{res}\mathfrak{N}_p$ iff $G \in \mathrm{res}\mathfrak{N}_p$ and $\bar{N} \in \mathrm{res}\mathfrak{N}_p$, where $\bar{N} = N/V(N)$.*

The necessity of the condition $G \in \mathrm{res}\mathfrak{N}_p$ was proven by Andreev and Olshanskii in [1]; the necessity of the condition $\bar{N} \in \mathrm{res}\mathfrak{N}_p$ is obvious.

The problem when the group $\bar{S}$ belongs to the class $\mathrm{res}\mathfrak{N}_p$ is mentioned in Hartley's exposition article [8] (see [8], the discussion after Problem 3.3.10). Our Corollary of Theorem A completely answers this question.

Our Theorem A can be considered as the analog of Šmelkin's result [18], which states that if the groups G and H are residually torsion free nilpotent then so is the verbal wreath product $H_{\mathrm{wr}_{\mathfrak{U}}} G$. Since we work with restricted Lie algebras and in particular don't make use of the Campbell–Hausdorff formulae, our methods are different from the ones used in [18]. It is important also to point out that any one of the three following properties of groups — nilpotency, torsion free nilpotency or being a nilpotent p-group of bounded exponent — is not a "star property" because it is not preserved by group extensions (see Neumann [15]) and because of this some powerful theorems of the theory of varieties of groups cannot be applied for the problems related to the residuality by these classes of groups.

Finally, our Theorem B can be considered as an analog of Hartley's results [6] on the residual nilpotence of discrete wreath product. Brian Hartley was one of the key innovators and leaders in the study of infinite groups and group rings; the residual nilpotence of groups and augmentation ideals of group rings was one of the many topics where his results were fundamental. This paper is dedicated to his memory.

2. Preliminaries

2.1 We recall first the definition of the verbal wreath product of groups and its properties.

Let $\mathfrak{U}$ be a variety of groups, H a free group of this variety, and G an arbitrary group. Then the group $F = H_{\mathrm{wr}_{\mathfrak{U}}} G$ is the verbal wreath product of H and G if F is generated by G and H and the following two properties holds:

(i) The normal closure of H in G belongs to the variety $\mathfrak{U}$. This normal closure $\bar{H}$ is in fact a relatively free group which is called the base group of the wreath product and F is a split extension of $\bar{H}$ by G.

(ii) If $\varphi\colon G \longrightarrow G_1$ and $\psi\colon H \longrightarrow H_1$ are homomorphisms into a group F_1 such that F_1 is generated by G_1 and H_1 and the normal closure of H_1 in F_1 belongs to $\mathfrak{U}$, then these homomorphisms are extended to a homomorphism $\theta\colon F \longrightarrow F_1$.

The second way to define the wreath product $H_{\mathrm{wr}_{\mathfrak{U}}} G$ is to take in the free product $G * H$ the normal closure H^* of H and to observe that $H_{\mathrm{wr}_{\mathfrak{U}}} G$ is isomorphic to the quotient group $(G * H)/V(H^*)$.

We will use throughout the paper the following well-known and simple fact:

Let G be a subgroup of G. Then the subgroups G_1 and H generate in F a subgroup F_1 isomorphic to the verbal wreath product $(G_1)_{\mathrm{wr}_{\mathfrak{U}}} H$.

2.2 We give now a brief account of some concepts and known results on the correspondence between groups and Lie algebras; the reader is referred to Lazard's fundamental article [10] for details. Let G be a group; a series of normal subgroups

$$G = G_1 \supseteq G_2 \supseteq \cdots \tag{2.1}$$

is an N_p-series if $(G_i, G_j) \supseteq G_{i+j}$ and for every $x \in G_i$ we have $x^p \in G_{ip}$. The series (2.1) defines a weight function on G: $w(x) = i$ if $x \in G_i \setminus G_{i+1}$ and $w(x) = \infty$ if $x \in \bigcap_{i=1}^{\infty} G_i$. The restricted Lie algebra associated to the series (2.1) is a Lie algebra over Z_p whose vector space is $\sum_{i=1}^{\infty} G_i/G_{i+j}$; if $w(g) = i$, then $\tilde{g} = gG_{i+1}$ is the homogeneous component of g; if $w(g) = \infty$, then $\tilde{g} = 0$. The Lie operation for homogeneous elements is defined in the following way. If $\tilde{x}_1 \in G_{i_1}/G_{i_1+1}$, $\tilde{x}_2 \in G_{i_2}/G_{i_2+1}$ then $[\tilde{x}_1, \tilde{x}_2]$ is the element of $G_{i_1+i_2}/G_{i_1+i_2+1}$ which contains the commutator (x_1, x_2), where x_α is the coset representative of $\tilde{x}_\alpha$ ($\alpha = 1, 2$); this operation is extended by distributivity on arbitrary elements from $\sum_{i=1}^{\infty} G_i/G_{i+1}$ and we obtain on the set $\sum_{i=1}^{\infty} G_i/G_{i+1}$ a structure of a graded Lie algebra. If $\tilde{x} \in G_i/G_{i+1}$ and x is its representative in G, then $(\operatorname{ad} x)^p = \operatorname{ad}(\tilde{x}^{[p]})$, where $\tilde{x}^{[p]} = (x)^p G_{ip+1}$ and we obtain the structure of a restricted algebra via the map $\tilde{x} \longrightarrow \tilde{x}^{[p]}$. We will denote this algebra, associated to the series (2.1), by $L_p(G, G_i)$. The most important example of a N_p-series is the series of dimension subgroups $M_i(G)$ ($i = 1, 2, \ldots$); we denote in this case the associated Lie algebra by $L_p(G)$. It is worth remarking that if (2.1) is an arbitrary N_p-series, then $M_i(G) \subseteq G_i$ ($i = 1, 2, \ldots$) and $G \in \mathrm{res}N_p$ iff $\bigcap_{i=1}^{\infty} M_i(G) = 1$.

It is known (see [16]) that the series (2.1) defined a filtration in the group ring KG (K is an arbitrary field of characteristic p):

$$A_0 = KG, \quad A_n = \mathrm{Sp}\left\{(x_{\alpha_1} - 1)(x_{\alpha_2} - 1) \cdots (x_{\alpha_s} - 1) \mid \sum_{i=1}^{s} w(x_{\alpha_i}) \geq n\right\} \tag{2.2}$$
$$(n = 1, 2, \ldots).$$

Let V be a subgroup G. The N_p-series (2.1) induces in V an N_p-series $V \cap G_i = V_i$ ($i = 1, 2, \ldots$); we denote the associated Lie algebra of V by $L_p^*(V, V_i)$. If V is a normal subgroup of G, then let $\bar{G}_i$ be the image of G_i in the quotient group $\bar{G} = G/V$ and $L_p(\bar{G}, \bar{G}_i)$ the Lie algebra of $\bar{G}$, associated to the N_p-series $\bar{G}_i$ ($i = 1, 2, \ldots$); in particular, when $G_i = M_i(G)$ we denote the Lie algebra $L_p^*(V, V_i)$ by $L_p^*(V)$. If $\varphi\colon G \longrightarrow \bar{G}$ is a homomorphism of groups, then it defines in a natural way a homomorphism of graded Lie algebras $\tilde{\varphi}\colon L_p(G) \longrightarrow L_p(\bar{G})$ (see Lazard [10]). We will make use of the following simple fact:

LEMMA 2.1: *Let G be a group which belongs to the class* $\mathrm{res}N_p$. *Consider a homomorphism $\varphi\colon G \longrightarrow \bar{G}$ and a related homomorphism of Lie algebras $\tilde{\varphi}\colon L_p(G) \longrightarrow L_p(\bar{G})$. Let g be an arbitrary non-unit element of G. Then:*

(i) *$\tilde{\varphi}(\tilde{g}) = 0$ iff $w(\varphi(g)) > w(g)$.*

(ii) *If $w(\varphi(g)) = w(g)$ then $\tilde{\varphi}(\tilde{g}) = \widetilde{\varphi(g)}$.*

Proof: Follows easily from the definition of the homomorphism $\tilde{\varphi}$. ∎

The proof of the following proposition does not differ essentially from the proof of Theorem 2.4 in Lazard's article [10] (see also [2], §2.3, 2.4).

PROPOSITION 2.1: *Let*

$$1 \longrightarrow V \xrightarrow{\alpha} G \xrightarrow{\beta} G/V \longrightarrow 0 \tag{2.3}$$

where α is the canonical injection $V \subseteq G$ and β is the natural homomorphism $G \xrightarrow{\beta} G/V$.

Then the sequence (2.3) defined in a natural way an exact sequence

$$0 \longrightarrow L_p^*(V, V_i) \longrightarrow L_p(G, G_i) \longrightarrow L_p(\bar{G}, \bar{G}_i) \longrightarrow 0 \tag{2.4}$$

and in particular, when $G_i = M_i(G)$,

$$0 \longrightarrow L_p^*(V) \longrightarrow L_p(G) \longrightarrow L_p(G/V) \longrightarrow 0. \tag{2.4'}$$

We will need also the following known fact:

LEMMA 2.2: *Let G be a group and g_j $(j \in J)$ be a system of elements which generates G. Let $w(g_j) = n_j$ $(j \in J)$. Then the elements $\tilde{g}_j = g_j + M_{n_j+1}(G)$ $(j \in J)$ generate $L_p(G)$.*

Proof: We pick a subset g_j $(j \in J_1)$ which gives a basis of the vector space $G/M_2(G)$ (and hence $w(g_j) = 1$ $(j \in J_1)$); it is well known that this system of elements $g_j + M_2(G)$ $(j \in J_1)$ generates $L_p(G)$ and the assertion follows.

LEMMA 2.3: *Let G be an arbitrary group, $L_p(G, G_i)$ the p-algebra of G, associated to the N_p-series (2.1), and $x_1, x_2, \dots, x_s$ be given elements of G. Assume that $\bigcap_{i=1}^{\infty} G_i = 1$. Then:*

(i) *If*

$$\begin{aligned} &w\left((x_1^{p^{m_1}}, x_2^{p^{m_2}}, \dots, x_s^{p^{m_s}})^{p^m}\right) > i_0 = \\ &= p^m(p^{m_1}w(x_1) + p^{m_2}w(x_2) + \dots + p^{m_s}w(x_s)), \end{aligned} \tag{2.5}$$

then in $L_p(G, G_i)$

$$\left[\tilde{x}_i^{[p]^{m_1}}, \tilde{x}_2^{[p]^{m_2}}, \dots, \tilde{x}_s^{[p]^{m_s}}\right]^{[p]^m} = 0. \tag{2.6}$$

(ii) *Let*

$$\left(x_1^{p^{m_1}}, x_2^{p^{m_2}}, \dots, x_s^{p^{m_s}}\right)^{p^m} \neq 1; \tag{2.7}$$

then the relation

$$\left(\left(x_1^{p^{m_1}}, x_2^{p^{m_2}}, \dots, x_s^{p^{m_s}}\right)^{p^m}\right) = \left[\tilde{x}_1^{[p]^{m_1}}, \tilde{x}_2^{[p]^{m_2}}, \dots, \tilde{x}_s^{[p]^{m_s}}\right]^{[p]^m} \tag{2.8}$$

holds in $L_p(G, G_i)$ iff

$$\begin{aligned} &w\left((x_1^{p^{m_1}}, x_2^{p^{m_2}}, \dots, x_s^{p^{m_s}})^{p^m}\right) \\ &= p^m\left(p^{m_1}w(x_1) + p^{m_2}w(x_2) + \dots + p^{m_s}w(x_s)\right). \end{aligned} \tag{2.9}$$

Proof: (i) The definition of $L_p(G, G_i)$ shows that the element in the left side of (2.6) is the coset

$$(x_1^{p^{m_1}}, x_2^{p^{m_2}}, \ldots, x_s^{p^{m_s}})^{p^m} G_{i_0}. \tag{2.10}$$

We have, however, from (2.5)

$$(x_1^{p^{m_1}}, x_2^{p^{m_2}}, \ldots, x_s^{p^{m_s}})^{p^m} \in G_{i_0},$$

and the assertion follows.

(ii) Once again as in (i) the left side of (2.8) is the coset of (2.10); and hence (2.8) follows immediately if (2.9) holds. If now (2.9) does not hold, then (2.5) holds and we obtain (2.6). But the left side of (2.8) cannot be zero because of (2.7) and the condition $\bigcap_{i=1}^{\infty} G_i = 1$. This completes the proof.

3. Restricted Lie algebras of finite p-groups

3.1 We assume in this section that G is a group with an N_p-series (2.1). We recall now that a basis for the algebra $L_p(G, G_i)$ can be constructed in the following way. For every given k we pick a basis E_k of the vector space G_k/G_{k+1} which contains a basis E'_k of the subspace $(G_k \cap V)/(G_{k+1} \cap V)$ (if it is non-zero). Let $E = \bigcup_{k=1}^{\infty} E_k$, $E' = \bigcup_{k=1}^{\infty} E'_k$. Then E is a basis for $L_p(G, G_i)$ and E' is a basis for $L_p^*(V, V_i)$. A weight function and an ordering on E are defined in a routine way: the elements of E_k have weight k and E_k is ordered in an arbitrary way; the elements of E_k precede the elements of E_{k+1}. The weight function $w(x)$ is extended in a natural way to the set of the basic monomials of the form

$$\begin{gathered} \pi = e_{j_1}^{n_1} e_{j_2}^{n_2} \cdots e_{j_k}^{n_k} \\ (j_1 < j_2 < \cdots < j_k; \quad 1 \le n_\alpha \le p-1; \quad \alpha = 1, 2, \ldots, k) \end{gathered} \tag{3.1}$$

and then to the p-envelope $U_p(L_p(G, G_i))$.

Let $\operatorname{gr}(Z_pG)$ be the graded ring of Z_pG associated to the filtration $\varpi^k(Z_pG)$ ($k = 0, 1, \ldots$); we recall that Quillen proved that the graded algebras $U_p(L_p(G))$ and $\operatorname{gr}(Z_pG)$ are isomorphic (see [17]); we will need the following modification of this result.

PROPOSITION 3.1: *For an arbitrary element $g \in E_k$ define a map*

$$gG_{k+1} \longrightarrow (g-1) + A_{k+1}. \tag{3.2}$$

Then this map is extended in a natural way to an isomorphism between the graded algebras $U_p(L_p(G, G_i))$ and the graded ring $\mathrm{gr}(Z_pG)$, associated to the filtration (2.2); furthermore, the restriction of the map (3.2) on $U_p(L_p^*(V))$ defines an isomorphism between the graded subalgebra $U_p(L_p^*(V))$ and the graded subalgebra $\mathrm{gr}^*(Z_pV) \subseteq \mathrm{gr}(Z_pG)$, associated to the filtration $Z_pV \cap A_k$ $(k = 1, 2, \ldots)$.

The fact that this map is an isomorphism is proved in [13], Theorem 2.4; the proof of the second statement can be read from [13] and we omit it.

3.2 If G is an arbitrary group,then its nth dimension subgroup $M_n(G)$ is defined by

$$M_n(G) = G \cap \varpi^n(Z_pG) \tag{3.3}$$

where $\varpi(Z_pG)$ is the augmentation ideal of Z_pG. It is known (see [16], Lemma 3.8) that if G is a finite p-group such that $M_{c+1}(G) = 1$ and Q is a subgroup of G, then the following important relation holds for all n:

$$(\varpi^{nc}(Z_pG)) \cap Z_pQ \subseteq \varpi^n(Z_pQ). \tag{3.4}$$

Now let L be an arbitrary restricted Lie algebra. The dimension subalgebras of L are defined by

$$M_n(L) = L \cap \varpi^n(U_p(L)) \quad (u = 1, 2, \ldots,), \tag{3.5}$$

where $\varpi(U_p(L))$ is the augmentation ideal of $U_p(L)$.

We apply now relation (3.4) together with Proposition 3.1 and obtain the following fact:

PROPOSITION 3.2: *Let G be a finite p-group and Q be a subgroup of G. Assume that $M_{c+1}(G) = 1$ and let $L_p(G)$ be the restricted Lie algebra of G associated to the series of dimension subgroups $M_i(G)$ $(i = 1, 2, \ldots)$. Then*

$$\varpi^{nc}(U_p(L_p(G))) \cap U_p(L_p^*(Q)) \subseteq \omega^n(U_p(L_p^*(Q))) \quad (n = 1, 2, \ldots). \tag{3.6}$$

Remark: Proposition 3.2 remains true for the case when L is an arbitrary restricted Lie algebra such that $M_{c+1}(L) = 0$; if Q is a restricted Lie subalgebra of L, then the following relation holds:

$$\varpi^{nc}(U_p(L)) \cap U_p(Q) \subseteq \varpi^n(U_p(Q)). \tag{3.7}$$

We considered in Proposition 3.2 the case when L is a restricted Lie algebra of a group, because we need only this case in our further arguments and the proof of relation (3.6) is simpler than the proof of (3.7).

We will need also the following simple fact:

LEMMA 3.1: *Let G be a finite p-group,* (2.1) *an arbitrary N_p-series in G and $L_p(G, G_i)$ the restricted Lie algebra associated to the series* (2.1). *Assume that*

$$\varpi^{n+1}(Z_pG) = 0. \tag{3.8}$$

Then

$$\varpi^{n+1}(U_p(L_p(G, G_i))) = 0. \tag{3.9}$$

Proof: Let $\mathrm{gr}(Z_pG)$ be the graded ring of Z_pG associated to the filtration (2.2). We obtain immediately from (3.8)

$$\varpi^{n+1}(\mathrm{gr}(Z_pG)) = 0 \tag{3.10}$$

and (3.9) now follows from (3.10) and Proposition 3.1.

COROLLARY: *Assume that the conditions of Lemma* 3.1 *hold. Then*

$$M_{n+1}(L_p(G, G_i)) = 0. \tag{3.11}$$

4. Restricted Lie algebras of verbal wreath products of groups

4.1 Let $F = H_{\mathrm{wr}_{\mathfrak{U}}} G$ and $\bar{H}$ be the base group of the wreath product. We begin now to study the Lie algebra $S = L_p(F)$. We first need the following simple fact:

LEMMA 4.1: *The subalgebra $L_p^*(G)$ of $L_p(F)$ is isomorphic to $L_p(G)$ and $L_p(F)$ is a split extension of $L_p^*(\bar{H})$ by $L_p(G)$.*

Proof: Consider epimorphism $\varphi\colon F \longrightarrow G$ with kernel $\bar{H}$. Since G is a semidirect factor of F, we obtain easily that $M_i(F) \cap G = M_i(G)$ $(i = 1, 2, \ldots)$ and hence $L_p^*(G) \simeq L_p(G)$. Furthermore, we obtain by Lemma 2.1 a related homomorphism $\tilde{\varphi}\colon L_p(F) \longrightarrow L_p(G)$ with kernel $L_p^*(\bar{H})$ and the quotient algebra $L_p(F)/L_p^*(\bar{H})$ is isomorphic to $L_p(G)$, i.e. $L_p(F)$ is a split extension of $L_p^*(\bar{H})$ by $L_p(G)$. This completes the proof. ∎

Now let G_1 be a subgroup of G; we construct in the algebra $L_p(G)$ a basis by the procedure which was described in subsection 3.2. The system of the basic monomials (3.4) will be used in Proposition 4.1 to construct a system of generators for the ideal $L_p^*(\bar{H})$.

LEMMA 4.2: *Let $T = (H_p)_{\mathrm{wr}}G$ be a discrete wreath product, where H_p is an abelian group of exponent p with a basis h_i $(i \in I)$; let $\bar{H}_p$ be the base group of this wreath product and $(\bar{H}_p)_1$ the base group of the wreath product $T_1 = (G_1)_{\mathrm{wr}}H_p$. Then the algebra $L_p(T)$ is isomorphic to the split extension of the abelian algebra $L_p^*(\bar{H}_p)$ (with a trivial p-map $x \longrightarrow x^{[p]} = 0$) by the Lie algebra $L_p(G)$. Furthermore, for every given n the system of elements*

$$\tilde{h}_i, \left[\tilde{h}_i, \underbrace{e_{j_1}, e_{j_1}, \ldots, e_{j_1}}_{n_1}, \ldots, \underbrace{e_{j_k}, e_{j_k}, \ldots, e_{j_k}}_{n_k}\right]$$
$$(i \in I;\ j_1 < j_2 < \cdots < j_k;\ 1 \le n_\alpha \le p-1;\ \alpha = 1, 2, \ldots, k)$$

with

$$1 + \sum_{\alpha=1}^{k} n_\alpha w(e_{j_\alpha}) = n$$

forms a basis of the subspace $(M_n(T) \cap \bar{H}_p)/(M_{n+1}(T) \cap \bar{H}_p) \subseteq L^*(\bar{H}_p)$; the basis of the subspace $(M_n(T) \cap (\bar{H}_p)_1)/(M_{n+1}(T) \cap (\bar{H}_p)_1)$ is given by those monomials which contain only the elements from E'.

Proof: Let $\omega^n(Z_pG)$ be the nth power of the augmentation ideal of Z_pG. The conjugation in T induces in a natural way a structure of a Z_pG-module in the normal subgroup $\bar{H}_p$; this module is free with a basis h_i $(i \in I)$. It is well known (and can be verified easily) that the subgroup $M_n(T)$ is a split extension of the normal subgroup $\omega^n(Z_pG) \cdot \bar{H}_p$ by the subgroup $M_n(G)$; hence $M_n(T) \cap \bar{H}_p = \omega^n(Z_pG) \cdot \bar{H}_p$ and $M_n(T) \cap (\bar{H}_p)_1 = (\varpi^n(ZG) \cap ZG_1) \cdot \bar{H}_p$. This implies that the nth homogeneous component $(M_n(T) \cap \bar{H}_p)/(M_{n+1}(T) \cap \bar{H}_p)$ of the ideal $L_p^*(\bar{H}_p)$ is isomorphic to the quotient space $(\omega^n(Z_pG) \cdot \bar{H}_p)/(\omega^{n+1}(Z_pG) \cdot \bar{H}_p)$ and $(M_n(T) \cap (\bar{H}_p)_1)/(M_{n+1}(T) \cap (\bar{H}_p)_1)$ is isomorphic to the quotient space $((\varpi^n(Z_pG) \cap ZG_1) \cdot H_p)/(\varpi^{n+1}(Z_pG) \cap ZG_1) \cdot H_p$; we obtain from this that the graded ideal $L_p^*(\bar{H}_p)$ is isomorphic to a free module with a basis $\tilde{h}_i$ $(i \in I)$ over the associated graded ring $\mathrm{gr}(Z_pG)$ and the graded ideal $L_p^*((\bar{H}_p)_1)$ is isomorphic to a free module with a basis $\tilde{h}_i$ $(i \in I)$ over the graded ring $\mathrm{gr}^*(Z_pG_1)$, associated to the filtration $\varpi^n(Z_pG) \cap Z_pG_1$ $(n = 1, 2, \ldots)$. Finally, we apply Proposition 3.1 and the assertion follows. ∎

LEMMA 4.3: *Let H be a p-group (a restricted Lie algebra). Assume that $M_{n+1}(H) = 1$ $(M_{n+1}(H) = 0)$, for some natural n, and let x_i $(i \in I)$ be a system of elements which generates H module $M_2(H)$. Then this system of elements generates H.*

The proof can be obtained by induction on the length of the number n and we omit it.

We recall (see [16]) *that if x_i $(i \in I)$ is a system of generators which gives a basis of $H/M_2(H)$, then it is called an independent system of generators.*

PROPOSITION 4.1: *Let $F = H_{\mathrm{wr}_{\mathfrak{U}}} G$, where H is a finite relatively free p-group with an independent system of generators $h_1, h_2, \ldots, h_n$, G is a finite p-group, G_1 its subgroup, and E and E' the bases of $L_p(G)$ and $L_p^*(G_1)$, which were constructed in subsection 3.1. Let $S = L_p(F)$ and $\tilde{h}_i = h_i + M_2(S)$ $(i = 1, 2, \ldots, n)$. Then the elements*

$$\tilde{h}_i, \left[\tilde{h}_i, \underbrace{e_{j_1}, e_{j_1}, \ldots, e_{j_1}}_{n_1}, \underbrace{e_{j_2}, e_{j_2}, \ldots, e_{j_2}}_{n_2}, \ldots, \underbrace{e_{j_k}, e_{j_k}, \ldots, e_{j_k}}_{n_k}\right] \tag{4.1}$$
$$(i = 1, 2, \ldots, n;\ j_1 < j_2 \cdots < j_k;\ 1 \le n_\alpha \le p-1;\ \alpha = 1, 2, \ldots, k)$$

form an independent system of generators of the algebra $L_p^*(\bar{H})$. Furthermore, those elements (4.1) which contain only the basic elements from E' form an independent system of generators for the subalgebra $L_p^*(\bar{H}_1)$.

Proof: We consider the homomorphism φ: $H \longrightarrow H/M_2(H) = H_p$ which maps H on an abelian group of exponent p; a basis of this group is given by the system of elements $t_i = \varphi(h_i)$ $(i = 1, 2, \ldots, n)$. The homomorphism φ defines an epimorphism ψ of the verbal wreath product $F = H_{\mathrm{wr}_{\mathfrak{U}}} G$ on the ordinary (discrete) wreath product $T = (H_p)_{\mathrm{wr}} G$. We obtain also a related homomorphism of Lie algebra $\tilde{\psi}$: $L_p(F) \longrightarrow L_p(T)$. Let $\bar{H}_p$ be the base group of T. Lemma 4.2 now implies that the system of the elements

$$\tilde{t}_i = t_i + M_2(T), \left[\tilde{t}_i, \underbrace{e_{j_1}, e_{j_1}, \ldots, e_{j_1}}_{n_1}, \underbrace{e_{j_2}, e_{j_2}, \ldots, e_{j_2}}_{n_2}, \ldots, \underbrace{e_{j_k}, e_{j_k}, \ldots, e_{j_k}}_{n_k}\right] \tag{4.2}$$
$$(i = 1, 2, \ldots, n;\ j_1 < j_2 < \cdots < j_k;\ 1 \le n_\alpha \le p-1;\ \alpha = 1, 2, \ldots, k)$$

is a basis of the ideal $L_p^*(\bar{H}_p)$. We see now from (4.1) and (4.2) that the images of the elements (4.1) form a basis of the abelian restricted Lie algebra $\psi(L_p^*(H))$ (with trivial p-map $x^{[p]} = 0$). Since $\ker \tilde{\psi} \supseteq M_2(L_p^*(\bar{H}))$ we obtain that these elements are linearly independent modulo the ideal $M_2(L_p^*(\bar{H}))$. Because we have a finite number of elements (4.1) and this number coincides with the rank of the algebra $(L_p^*(\bar{H}))/M_2(L_p^*(\bar{H}))$ we conclude that in fact $\ker \tilde{\psi} = M_2(L_p^*(\bar{H}))$. We obtain now that the elements (4.1) generate the algebra $L_p^*(\bar{H})$ modulo the ideal $M_2(L_p^*(\bar{H}))$ and Lemma 4.3 implies that they generate $L_p^*(\bar{H})$. We proved that the elements (4.1) form an independent system of generators for $L_p^*(\bar{H})$.

To prove the second part of the assertion we observe that those generators (4.1) which contain only the basic elements from E' belong to $L_p^*(H_1)$ and Lemma 4.2 yields that they generate $L_p^*(\bar{H}_1)$ modulo the ideal $(M_2(L_p^*(\bar{H})) \cap L_p^*(\bar{H}_1)) \supseteq M_2(L_p^*(H_1))$; we have just proved that they are independent modulo this ideal. We conclude as above that $M_2(L_p^*(\bar{H})) \cap L_p^*(\bar{H}_1) = M_2(L_p^*(\bar{H}_1))$ and hence the generators which include only the basic elements from E' form an independent system of generators for $L_p^*(\bar{H}_1)$. This completes the proof. ∎

Remark: Proposition 4.1 and all the further results in Sections 4 and 5 remain true in the general case, when $G \in \mathrm{res}\mathfrak{N}_p$, $H \in \mathrm{res}\mathfrak{N}_p$ (and H is relatively free). We consider the case when G and H are finite because this is sufficient for the proof of Theorem A.

4.2 We need now the "multiplicative analogs" of elements (4.1). For every element $e_j \in M_{k_j}(G)/M_{k_j+1}(G)$ which occurs in (4.1), we take its representative e_j^* in $M_{k_j}(G)$; if e_j belongs to $L_p^*(G_1)$, we take its representative in $M_{k_j}(G) \cap G_1$. We then consider the following multiplicative commutators in the group $F = H_{\mathrm{wr}_\mathfrak{U}} G$:

$$(4.1') \quad \Big(h_i, \big(h_i, \underbrace{e_{j_1}^*, e_{j_1}^*, \ldots, e_{j_1}^*}_{n_1}, \underbrace{e_{j_2}^*, e_{j_2}^*, \ldots, e_{j_2}^*}_{n_2}, \ldots, \underbrace{e_{j_k}^*, e_{j_k}^*, \ldots, e_{j_k}^*}_{n_k}\big)\Big)$$
$$(i = 1, 2, \ldots, n;\ j_1 < j_2 < \cdots < j_k;\ 1 = n_\alpha \leq p - 1;\ \alpha = 1, 2, \ldots, k).$$

LEMMA 4.4: *Assume that the conditions of Proposition 4.1 hold. Then the elements (4.1′) form an independent system of generators for the base group $\bar{H}$ whereas those elements which include only representatives from G_1 form a free system of generators for the group $\bar{H}_1$. Furthermore, the weights of the elements (4.1′) with respect to the filtration*

$$F = M_1(F) \supseteq M_2(F) \supseteq \cdots$$

are

$$w(h_i) = 1;\ w\Big(h_i, \underbrace{e_{j_1}^*, e_{j_1}^*, \ldots, e_{j_1}^*}_{n_1}, \underbrace{e_{j_2}^*, e_{j_2}^*, \ldots, e_{j_2}^*}_{n_2}, \ldots, \underbrace{e_{j_k}^*, e_{j_k}^*, \ldots, e_{j_k}^*}_{n_k}\Big)$$
$$(4.3) \quad = 1 + n_1 w(e_{j_1}) + n_2 w(e_{j_2}) + \cdots + n_k w(e_{j_k})$$
$$(i = 1, 2, \ldots, n;\ j_1 < j_2 < \cdots < j_k;\ 1 \leq n_\alpha \leq p - 1;\ \alpha = 1, 2, \ldots, k).$$

Proof: Once again, as in the proof of Proposition 4.1, we consider the homomorphism $\varphi\colon H_{\mathrm{wr}_{\mathfrak{U}}} G \longrightarrow (H_p)_{\mathrm{wr}} G$, where $H_p = H/M_2(H)$. We apply the same argument and obtain the first two statements of the assertion. To prove the relation (4.3) we observe first that it is obvious that

$$w\left(h_i, \underbrace{e^*_{j_1}, e^*_{j_1}, \ldots, e^*_{j_1}}_{n_1}, \underbrace{e^*_{j_2}, e^*_{j_2}, \ldots, e^*_{j_2}}_{n_2}, \ldots, \underbrace{e^*_{j_k}, e^*_{j_k}, \ldots, e^*_{j_k}}_{n_k}\right) \\ \geq 1 + n_1 w(e_{j_1}) + n_2 w(e_{j_2}) + \cdots + n_k w(e_{j_k}).$$

So, in order to establish the relation (4.3) it is enough to show that it holds in a homomorphic image of F. But Lemma 4.2 implies that it holds in the group $(H_p)_{\mathrm{wr}} G$ and the proof is completed. ∎

COROLLARY 1: *The elements* (4.1) *are the homogeneous components in* $L_p(F)$ *of the elements* (4.1$'$). *Hence*

$$w(\tilde{h}_i) = 1; w\left[\tilde{h}_i, \underbrace{e^*_{j_1}, e^*_{j_1}, \ldots, e^*_{j_1}}_{n_1}, \underbrace{e^*_{j_2}, e^*_{j_2}, \ldots, e^*_{j_2}}_{n_2}, \ldots, \underbrace{e^*_{j_k}, e^*_{j_k}, \ldots, e^*_{j_k}}_{n_k}\right] \\ = 1 + n_1 w(e_{j_1}) + n_2 w(e_{j_2}) + \cdots + n_k w(e_{j_k}). \tag{4.4}$$

Proof: Follows from Lemmas 4.4 and 2.5. ∎

COROLLARY 2: *Let* $S = L_p(F)$, $S_1 = L^*_p(F_1)$ *where* $F_1 = H_{\mathrm{wr}} G_1$. *Let* $x \neq \tilde{h}_i$ *be an element from the system of generators* (4.1). *Assume that* $x \in S_1$ *and* $w_S(x) \geq nc$. *Then* $w_{S_1}(x) \geq n$.

Proof: Formula (4.4) implies that

$$e^{n_1}_{j_1} e^{n_2}_{j_2} \cdots e^{n_k}_{j_k} \in \varpi^{nc}(\mathfrak{U}_p(L_p(G))).$$

We conclude now from Proposition 3.2 that

$$e^{n_1}_{j_1} e^{n_2}_{j_2} \cdots e^{n_k}_{j_k} \in \varpi^{n}(\mathfrak{U}_p(L_p(G_1)))$$

and hence

$$n_1 w_{S_1}(e_{j_1}) + n_2 w_{S_1}(e_{j_2}) + \cdots + n_k w_{S_1}(e_{j_k}) \geq n$$

and the assertion follows. ∎

4.3. LEMMA 4.5: *Let*

$$u_1 = [x_{\alpha_1}, x_{\alpha_2}, \dots, x_{\alpha_{r_1}}]^{[p]^{m_1}}, \quad u_2 = [x_{\beta_1}, x_{\beta_2}, \dots, x_{\beta_{r_2}}]^{[p]^{m_2}},$$
$$u_k = [x_{\gamma_1}, x_{\gamma_2}, \dots, x_{\gamma_{r_k}}]^{[p]^{m_k}}$$

be Lie monomials in generators (4.1). (We do not assume that these monomials are different.) Assume that

$$p^{m_1}\left(\sum_\alpha w(x_\alpha)\right) = n_1, \quad p^{m_2}\left(\sum_\beta w(x_\beta)\right) = n_2, \quad \dots,$$
$$p^{m_k}\left(\sum_\gamma w(x_\gamma)\right) = n_k.$$

Then the Lie product $[u_1, u_2, \dots, u_k]$ is a sum of Lie monomials of the form $[x_{\delta_1}, x_{\delta_2}, \dots, x_{\delta_r}]^{[p]^l}$ satisfying the condition

$$p^l\left(\sum_\delta w(x_\delta)\right) \ge n_1 + n_2 + \cdots + n_k.$$

Proof: Clearly, it is enough to consider the case $k = 2$. The assertion is obvious when $m_1 = m_2 = 0$; to obtain it in the general case we combine the following two identities:

$$\left[u, v^{[p]^{m_2}}\right] = \Bigg[u, \underbrace{v, v, \dots, v}_{p^{m_2}}\Bigg]$$

and

$$\left[u^{[p]^{m_1}}, v\right] = -\left[v, u^{[p]^{m_1}}\right] = -\Bigg[v, \underbrace{u, u, \dots, u}_{p^{m_1}}\Bigg],$$

which are true in an arbitrary restricted Lie algebra. ∎

LEMMA 4.6: *Let $S = L_p(F)$, x be an arbitrary element from the system of generators (4.1) and e be an element from $L = L_p(G)$; assume that $w(x) = n$, i.e. $x \in M_n(S) \setminus M_{n+1}(S)$. Then $[x, e]$ belongs to the subalgebra of $L_p^*(\bar{H})$ generated by the generators with weights greater than or equal to $n+1$.*

Proof: Clearly, we can assume that e is an element of the basis of $L_p(G)$; since the case when $x = h_i$ is trivial, we can consider the case when x has the form

$$x = \Bigg[h_i, \underbrace{e_{j_1}, e_{j_1}, \dots, e_{j_1}}_{n_1}, \underbrace{e_{j_2}, e_{j_2}, \dots, e_{j_2}}_{n_2}, \dots, \underbrace{e_{j_k}, e_{j_k}, \dots, e_{j_k}}_{n_k}\Bigg]$$
$$(j_1 < j_2 < \cdots < j_k;\ 1 \le n_\alpha \le p-1;\ \alpha = 1, 2, \dots, k).$$

In order to represent the element $[x, e]$ through the generators (4.1), we can apply the standard method similar to the proof of the Poincare–Birkhoff–Witt Theorem:

If $e < e_{j_k}$, we apply the identity $[[a,b],e] = [a,[b,e]] + [[a,e],b]$; if $e = e_{j_k}$ but $n_k = p-1$, we apply the identity

$$[a, \underbrace{e, e, \ldots, e}_{p}] = [a, e^{[p]}].$$

It is easy to see that all the generators which will be obtained at the end of the induction process will have weight greater than or equal to $n+1$. The proof can be completed easily. ∎

LEMMA 4.7: *Let $u = [x_1, x_2, \ldots, x_r]^{[p]^m}$ be a Lie monomial in the generators (4.1). Assume that*

$$p^m \left(\sum_{\alpha=1}^{r} w(x_\alpha) \right) = n \tag{4.5}$$

and let e be an arbitrary element from $M_k(L)$ $(k \geq 1)$. Then $[u, e]$ is a sum of monomials of the form $[x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_\beta}]^{[p]^l}$ satisfying the condition

$$p^l \left(\sum_{\beta=1}^{s} w(x_{\alpha_\beta}) \right) \geq n + k. \tag{4.6}$$

Proof: We consider first the case when $m = 0$ and $k = 1$. Then in this case Lemmas 4.5 and 4.6 together with the relation

$$\begin{aligned} [[x_1, x_2, \ldots, x_r], e] =& [[x_1, e], x_2, \ldots, x_r] + [x_1, [x_2, e], \ldots, x_r] + \cdots \\ & + [x_1, x_2, \ldots, [x_r, e]] \end{aligned} \tag{4.7}$$

yield the assertion. We consider now the second case when $m \geq 1$ and $k = 1$. We apply in this case the relation

$$\begin{aligned} [u, e] &= -[e, [x_1, x_2, \ldots, x_r]^{[p]^m}] \\ &= -\left[e, \underbrace{[x_1, x_2, \ldots, x_r], [x_1, x_2, \ldots, x_r], \ldots, [x_1, x_2, \ldots, x_r]}_{p^m} \right] \\ &= \left[[[x_1, x_2, \ldots, x_r], e], \underbrace{[x_1, x_2, \ldots, x_r], \ldots, [x_1, x_2, \ldots, x_r]}_{p^m - 1} \right. \end{aligned} \tag{4.8}$$

and the assertion follows from (4.8) together with the case when $m = 0$ which has just been considered and Lemma 4.5.

Finally, consider the general case when $k > 1$. We apply induction by k. We can assume without loss of generality that the element e is either a form of $[a, b]$, $a \in M_{k-1}(L)$, $b \in L$, or $e = c^{[p]}$, $c \in M_{k_1}(L)$, where k_1 is the smallest number such that $pk_1 \geq k$. In the first case we have

$$[u, e] = [u, [a, b]] = [[u, a], b] - [[u, b], a]; \tag{4.9}$$

in the second case

$$[u, e] = [u, c^{[p]}] = \left[u, \underbrace{c, c, \ldots, c}_{p} \right] \tag{4.10}$$

and the induction is completed easily via the use of (4.9) and (4.10) and Lemmas 4.5 and 4.6. ∎

Proposition 4.2: *Assume that the conditions of Proposition 4.1 hold. Let y be an arbitrary element of $S = L_p(F)$. Then $y \in M_n(S)$ iff $y = y_1 + y_2$, where $y_1 \in M_n(L)$, $y_2 \in L_p^*(\bar{H})$, and is a linear combination of Lie monomials of the form $[x_1, x_2, \ldots, x_r]^{[p]^m}$ where*

$$p^m \left(\sum_{\alpha=1}^{r} w(x_\alpha) \right) \geq n. \tag{4.11}$$

Proof: Since S is a split extension of $L_p^*(\bar{H})$ by L we obtain immediately that $y \in M_n(S)$ iff $y = y_1 + y_2$, where $y_1 \in M_n(L)$, $y_2 \in L_p^*(\bar{H})$. Proposition 4.1 implies that y_2 is a linear combination of Lie monomials of the form $[x_1, x_2, \ldots, x_r]^{[p]^m}$ where x_α $(\alpha = 1, 2, \ldots, r)$ is an element from the system of generators (4.1). If $w(x_\alpha) = n_\alpha$ $(\alpha = 1, 2, \ldots, r)$ then we obtain from condition (4.11) that

$$[x_1, x_2, \ldots, x_r]^{[p]^m} \subseteq [M_{n_1}(S), M_{n_2}(S), \ldots, M_{n_r}(S)]^{[p]^m} \subseteq M_n(S)$$

and we have proved the sufficiency of the conditions of the assertion.

To prove the necessity of these conditions we use induction by n. For the case $n = 1$ the assertion follows from Proposition 4.1. Assume that the assertion is proven for all terms $M_\alpha(S)$ $(\alpha = 1, 2, \ldots, n)$. Now let $y \in M_{n+1}(S)$. Since

$$M_{n+1}(S) = [M_n(S), S] + (M_{n_1}(S))^{[p]}, \tag{4.12}$$

where n_1 is the smallest integer satisfying $pn_1 \geq n+1$, we have to consider two cases:

(1) $y = [z, s]$, $z \in M_n(S)$, $s \in S$,

(2) $y = z^{[p]}$, $z \in M_{n_1}(S)$ where $n_1 p \geq n+1$.

In the first case we can assume that $z = u + v$, where $v \in M_n(L)$ and u is a sum of monomials of the form $[x_1, x_2, \ldots, x_r]^{[p]^m}$ satisfying (4.11). Let $s = s_1 + s_2$, $s_1 \in L$, $s_2 \in L_p^*(\bar{H})$. Then

$$[z, s] = [u + v, s_1 + s_2] = [u, s_1] + [v, s_1] + [u, s_2] + [v, s_2]. \tag{4.13}$$

The summand $[v, s_1]$ in (4.13) belongs to $M_{n+1}(L)$; Lemma 4.7 implies that the summands $[u, s_1]$ and $[v, s_2]$ are the sum of monomials satisfying (4.6) (with $k = 1$). Finally, the required representation for $[u, s_2]$ is obtained from Lemma 4.5. This completes the study of case (1).

In case (2) when $y = z^{[p]}$ the induction assumption implies that z is a linear combination of monomials of the form $[x_1, x_2, \ldots, x_r]^{[p]^{m_1}}$ satisfying

$$p^{m_1} \left(\sum_{\alpha=1}^{r} w(x_\alpha) \right) \geq n_1 \quad (pn_1 \geq n+1). \tag{4.14}$$

Hence the pth power of such a monomial is $[x_1, x_2, \ldots, x_r]^{[p]^{m_1+1}}$ where

$$p^{m_1+1} \left(\sum_{\alpha=1}^{r} w(x_\alpha) \right) \geq pn_1 \geq n+1$$

and the proof is completed if $y = z^{[p]}$, where z is one monomial. If now z is a linear combination of monomials π_β ($\beta = 1, 2, \ldots, k$), say

$$z = \sum_{\beta=1}^{k} \lambda_\beta \pi_\beta \quad (\lambda_\beta \in K; \quad \beta = 1, 2, \ldots, k), \tag{4.15}$$

then we apply the identity which holds in every restricted Lie p-algebra (see [9], V.7):

$$(a+b)^{[p]} = a^{[p]} + b^{[p]} + \sum_{i=1}^{p-1} s_i(a, b)$$

where $s_i(a, b)$ is the coefficient of λ^{i-1} in $a(\mathrm{ad}(\lambda a + b))^{p-1}$; hence $s_i(a, b)$ is a sum of commutators of length p in the Lie algebra generated by a, b. This together with Lemma 4.5 makes it possible to apply the induction on k and to obtain finally that $z^{[p]}$ is a sum of monomials of the form $[x_1, x_2, \ldots, x_r]^{[p]^m}$ satisfying

$$p^m \left(\sum_{\alpha=1}^{r} w(x_\alpha) \right) \geq n+1$$

and the assertion follows. ∎

COROLLARY: *Let f be an element of F. Then $f \in M_n(F)$ iff $f = gh$, where $g \in M_n(G)$ and $h \in H$, and is a product of elements of the form $[x_1^*, x_2^*, \ldots, x_r^*]^{p^m}$ where x_α^* are taken from the system of generators (4.1′) and*

$$p^m \sum_{\alpha=1}^{r} w(x_\alpha^*) \geq n. \tag{4.16}$$

Proof: Follows immediately from Proposition 4.2 together with Lemma 2.3. ∎

4.4 We consider now once again the independent system of generators (4.1′) for $\bar{H}$ which was obtained in Lemma 4.4. Let G_1 be a subgroup of G. We assume that G and H are finite p-groups. Take any of these generators, say an element x, and define $\varphi(x) = x$ if x contains only the representatives from G_1 and $\varphi(x) = 1$ for all the other generators. Since $\bar{H}$ is relatively free, this map is extended to an epimorphism $\bar{H} \longrightarrow \bar{H}_1$ where $\bar{H}_1$ is the base group of the verbal wreath product $F_1 = H_{\mathrm{wr}_{\mathfrak{U}}} G_1$; we denote this epimorphism also by φ. We can now describe φ as an epimorphism $\bar{H} \longrightarrow \bar{H}_1$ such that $\varphi(x) = x$ if the generator x belongs to $\bar{H}_1$ and $\varphi(x) = 1$ is x does not belong to $\bar{H}_1$. We recall (see Corollary of Lemma 4.4) that the generators (4.1) are homogeneous components of the elements (4.1′). We denote by $\tilde{x}$ the homogeneous component of the element x.

THEOREM 4.1: *The epimorphism φ defines in a natural way an epimorphism*

$$\tilde{\varphi}\colon L_p^*(\bar{H}) \longrightarrow L_p^*(\bar{H}_1)$$

such that it holds for every generator $\tilde{x}$ from the system (4.1):

$$\varphi(\tilde{x}) = \tilde{x} \quad \text{if } \tilde{x} \in L_p^*(\bar{H}_1), \tag{4.17}$$

$$\varphi(\tilde{x}) = 0 \quad \text{if } \tilde{x} \notin L_p^*(\bar{H}_1). \tag{4.18}$$

Proof: Assume that $h \in M_n(F) \cap \bar{H}$. Corollary of Proposition 4.2 implies that h is a product of commutators of the type $[x_1^*, x_2^*, \ldots, x^*]^{p^m}$ which satisfy (4.16). We have now $\varphi([x_1^*, x_2^*, \ldots, x_r^*]^{p^m}) = [x_1^*, x_2^*, \ldots, x_r^*]^{p^m}$ if $x_\alpha^* \in \bar{H}_1$ ($\alpha =$

$1, 2, \ldots, r$) and $\varphi([x_1^*, x_2^*, \ldots, x_r^*]^{p^m}) = 1$ if some of the x_α^* do not belong to $\bar{H}_1$. We obtain from this that $\varphi(x)$ is either 1 or a product of elements of the form $[x_1^*, x_2^*, \ldots, x_r^*]^{p^m}$ which satisfy condition (4.16). Corollary of Proposition 4.2 implies that $\varphi(x) \in (M_n(F) \cap \bar{H}) \cap \bar{H}_1 = M_n(F) \cap \bar{H}_1$. We have obtained that for every given n the epimorphism φ maps $M_n(F) \cap \bar{H}$ on $M_n(F) \cap \bar{H}_1$. This implies (see [10]) that φ induces a homomorphism $\tilde{\varphi}$: $L_p^*(\bar{H}) \longrightarrow L_p^*(\bar{H}_1)$ of the Lie algebras associated to the series $M_n(F) \cap \bar{H}$, $M_n(F) \cap \bar{H}_1$ ($n = 1, 2, \ldots,$) respectively. Finally, for every generator $x \in \bar{H}_1$ we have $\varphi(x) = x$ and hence $\tilde{\varphi}(\tilde{x}) = \tilde{x}$; this proves (4.17) and shows that $\tilde{\varphi}$ is an epimorphism. The relation (4.18) follows from the fact that $\varphi(x) = 1$ is $x \notin \bar{H}_1$. The proof is completed. ∎

COROLLARY: *The algebra $L_p^*(\bar{H}_1)$ is generated by those elements* (4.1) *which contain only the basic elements e_j from the subset $E' \subseteq G_1$.*

Proof: Follows immediately from Theorem 4.1. ∎

5. Restricted Lie algebras of subgroups of verbal wreath products

THEOREM 5.1: *Let $F = H_{\mathrm{wr}_{\mathfrak{U}}} G$. Assume that $M_{c+1}(G) = 1$. Let G_1 be a subgroup of G and $F_1 = H_{\mathrm{wr}_{\mathfrak{U}}} G_1$ (and hence $F_1 \subseteq F$). Let s be an element of $S_1 = L_p^*(F_1)$ and assume that $s \in (M_{n^2c}(S) \cap S_1)$. Then $s \in M_n(S_1)$.*

Proof: Once again we pick in the algebra $L = L_p(G)$ a basis which is obtained in Section 4 and then obtain a free system of generators for the algebra $L_p^*(\bar{H})$ and for the subalgebra $L_p^*(\bar{H}_1)$ (see Corollary of Proposition 4.2). An arbitrary element $s \in S$ has a unique representation

$$s = y_1 + y_2, \quad y_1 \in L, \quad y_2 \in L_p^*(\bar{H}). \tag{5.1}$$

If $s \in S_1$, then in (5.1), $y_1 \in L_1 = L_p^*(G_1)$. If now the element $s \in S_1$ belongs to $M_{n^2c}(S)$, then we obtain immediately that $y_1 \in M_{n^2c}(L) \cap L_1$ and Corollary of Proposition 3.2 implies that $y_1 \in M_n(L_1)$. Proposition 4.2 implies that y_2 is a sum of monomials of the form $[x_1, x_2, \ldots, x_r]^{[p]^m}$ satisfying the relation

$$p^m \left(\sum_{\alpha=1}^{r} w(x_\alpha) \right) \geq n^2 c; \tag{5.2}$$

and Corollary of Theorem 4.1 now implies that we can assume that the elements x_α ($\alpha = 1, 2, \ldots, r$) in (5.2) belong to the system of generators of $L_p^*(\bar{H}_1)$. Now if for some x_α we have $w_S(x_\alpha) \geq nc$, then we obtain from Corollary 2 of Lemma 4.3 that $w_{S_1}(x_\alpha) \geq n$ and Proposition 4.2 implies that $x_\alpha \in M_n(S_1)$. Hence, in this case $[x_1, x_2, \ldots, x_r] \in M_n(S_1)$. We can assume therefore that $w_S(x_\alpha) < nc$ ($\alpha = 1, 2, \ldots, r$) and hence we obtain from (5.2) that $p^m rnc \geq n^2 c$, i.e. $p^m r \geq n$. Since $w_{S_1}(x_\alpha) \geq 1$ ($\alpha = 1, 2, \ldots, r$) we obtain now

$$p^m \sum_{\alpha=1}^{r} w_{S_1}(x_\alpha) \geq n.$$

The last relation together with Proposition 4.2 implies that

$$\left([x_1, x_2, \ldots, x_r]^{[p]^m}\right) \in M_n(S_1),$$

which completes the proof. ∎

6. The proof of the main results: Theorem A

THEOREM A: *Let $F = H_{\mathrm{wr}_{\mathfrak{U}}} G$ be a verbal wreath product. Assume that $G \in \mathrm{res}\mathfrak{N}_p$, $H \in \mathrm{res}\mathfrak{N}_p$. Then $F \in \mathrm{res}\mathfrak{N}_p$.*

Proof: A routine argument reduces the proof to the case when $G \in \mathfrak{N}_p$, $H \in \mathfrak{N}_p$. Assume that $1 \neq f \in \bigcap_{k=1}^{\infty} M_k(F)$. We can find a finitely generated subgroup $H_0 \subseteq H$ which is a free factor in H such that $f \in \bigcap_{k=1}^{\infty} M_k(F_0)$, where $F_0 = (H_0)_{\mathrm{wr}_{\mathfrak{U}}} G$. There exists an endomorphism θ: $F \longrightarrow F_0$ such that

$$\theta(g) = g \quad (g \in G); \quad \theta(h) = h \quad (h \in H_0) \quad \text{and} \quad \theta(h) = 1 \quad \text{if } h \in H \setminus H_0.$$

We see that $\theta(f) = f$ and $f \in \bigcap_{k=1}^{\infty} M_k(F_0)$, i.e. we can assume that the group H is finitely generated and hence is finite.

Now find a finitely generated subgroup $G_1 \subseteq G$ such that $f \in F_1 = H_{\mathrm{wr}_{\mathfrak{U}}} G_1$. Since $G \in \mathfrak{N}_p$, $H \in \mathfrak{N}_p$ and these groups are finitely generated, we obtain that F_1 is a finite p-group. Assume that $M_{c+1}(G) = 1$ and that $\omega^{n+1}(Z_p F_1) = 0$. We take a finitely generated subgroup of G, $G_2 \supseteq G_1$, such that $f \in M_{(n+1)^2 c}(F_2)$ where $F_2 = H_{\mathrm{wr}_{\mathfrak{U}}} G_2$.

Now let $L_p(F_2)$ be the restricted Lie N_p-algebra of F_2 corresponding to the N_p-series $M_k(F_2)$ ($k = 1, 2, \ldots$); we denote by $\tilde{f}$ the homogeneous component of the element f. We obtain from Lemma 2.3 or by a straightforward argument that $\tilde{f} \in M_{(n+1)^2 c}(L_p(F_2))$.

Let $L_p^*(F_1)$ be the Lie algebra of F_1 which corresponds to the N_p-series $F_1 \cap M_k(F_2)$ ($k = 1, 2, \ldots$). Clearly, $L_p^*(F_1)$ is isomorphically imbedded into $L_p(F_2)$.

We apply now Theorem 5.1 and obtain that $\tilde{f} \in M_{n+1}(L_p^*(F_1))$. But Corollary of Lemma 3.1 implies that $M_{n+1}(L_p^*(F_1)) = 0$; in particular, $\tilde{f} = 0$. We see that the homogeneous component of f in the Lie algebra $L_p^*(F_2)$ is zero, which is impossible since F_2 is a finite p-group. This completes the proof. ∎

7. The proofs of the main results: Theorems B and C

7.1 Proof of the sufficiency of the conditions of Theorems B and C: In case (i) (of both theorems) Šmelkin's theorem [18] implies that F is residually torsion free nilpotent and hence $\bigcap_{k=1}^{\infty} \varpi^k(ZF) = 0$.

Condition (ii) of Theorem C includes as subcases condition (ii) of Theorem B or the subcase of condition (iii) of Theorem B (when all the orders are powers of the same prime number p); we obtain from Theorem A that condition (ii) of Theorem C yields that $F \in \mathrm{res}\mathfrak{N}_p$; it is known (see [6]) that this implies that $\bigcap_{k=1}^{\infty} \varpi^k(ZF) = 0$.

If the second subcase of condition (iii) in Theorem B holds then we see that F is in fact a discrete wreath product $F = G_{\mathrm{wr}}H$, where the abelian group H is a direct product of its prime components, $H = H_1 \times H_2 \times \cdots \times H_k$, and hence F is a subdirect product of groups $F_i = G_{\mathrm{wr}}H_i$ $(i = 1, 2, \ldots, k)$. For each of these groups F_i we have $F \in \mathrm{res}\mathfrak{N}_{p_i}$ $(i = 1, 2, \ldots, k)$ and hence $F \in \mathrm{res}\mathfrak{N}$.

We will now consider case (iii) of Theorem C and prove that in this case we obtain that $F \in \mathrm{dis}\, F_i$, $F_i \in \mathfrak{N}_{p_i}$ $(i \in I)$. This will imply the residual nilpotence of the ideal $\varpi(ZF)$ and will also prove case (iv) of Theorem B. We pick an arbitrary element $x \in F$. We have

$$x = gy, \tag{7.1}$$

where $g \in G$ and y has a representation

$$y = \bigcap_{\gamma} g_\gamma^{-1} h_\gamma g_\gamma \quad (g_\gamma \in G, h_\gamma \in H) \tag{7.2}$$

(some of the elements g_γ can be equal to 1). We obtain from this representation a system of non-unit elements (1.1) and find a prime number p_i such that none of these elements has an infinite p_i-height in G or H respectively. We hence obtain that for some kth term of the N_p-series of dimension subgroups

$$g_\alpha \notin M_k(G) \quad (\alpha = 1, 2, \ldots, r), \quad h_\beta \notin M_k(H) \quad (\beta = 1, 2, \ldots, s).$$

Let $\bar{G} = G/M_k(G)$, $\bar{H} = H/M_k(H)$, and $\bar{\mathfrak{U}}$ be the variety generated by the relatively free group $\bar{H}$. The homomorphisms $G \longrightarrow \bar{G}$ and $H \longrightarrow \bar{H}$ define a homomorphism φ of verbal wreath products

$$\varphi\colon G_{\mathrm{wr}_{\mathfrak{U}}} H \longrightarrow \bar{G}_{\mathrm{wr}_{\bar{\mathfrak{U}}}} \bar{H}$$

such that

$$\varphi(g_\alpha) = \bar{g}_\alpha \neq 1 \quad (\alpha = 1, 2, \ldots, r); \quad \varphi(h_\beta) = \bar{h}_\beta \neq 1$$

and hence

$$\varphi(x) = \bar{x} \neq 1.$$

Since $\bar{G} \in \mathfrak{N}_{p_i}$ and $\bar{H} \in \mathfrak{N}_{p_i}$ we obtain from Theorem A that the group $\bar{F} = \bar{G}_{\mathrm{wr}_{\mathfrak{U}}} \bar{H}$ belongs to the class $\mathrm{res}\mathfrak{N}_{p_i}$. We can find a homomorphism $\psi\colon \bar{F} \longrightarrow \tilde{F}$ such that $\tilde{F} \in \mathfrak{N}_p$ and $\psi(\bar{x}) \neq 1$. Finally, the homomorphism $\psi\varphi$ maps F on a nilpotent group $\tilde{F}$ and $\psi\varphi(x) \neq 1$; since x was an arbitrary non-unit element of F we have proved that F is residually nilpotent. This completes the proof of sufficiency of the conditions of Theorem B and Theorem C.

7.2 We recall first the following facts from Malcev's paper [14]:

LEMMA 7.1: *Let S be a residually nilpotent group. Then:*

(i) *The set of generalized periodic elements is a normal subgroup of S.*
(ii) *For a given p the set of all p-elements is a normal subgroup as well as the set of all the elements of infinite p-height.*
(iii) *An element of infinite p-height commutes with every p-element.*

It is not difficult to obtain from Lemma 7.1 the following fact (see [12]):

LEMMA 7.2: *Let S be a residually torsion free nilpotent group, s a generalized periodic element and r an element of S which has an infinite p_i-height for every $p_i = p_i(r)$. Then r commutes with s.*

Let $F = G_{\mathrm{wr}_{\mathfrak{U}}} H$; we recall that $F = (G * H)/(V(H^*))$. Now take the free product $G * H$ and its cartesian subgroup C. This cartesian subgroup is freely generated by all the elements $[g, h]$ $(1 \neq g \in G; 1 \neq h \in H)$ (see [3]); it is easy to obtain from this that it is a free factor in the normal subgroup H^*. More precisely,

$$H^* \simeq C * H. \tag{7.3}$$

We pass now to the verbal wreath product $F = (G * H)/(V(H^*))$. The relation (7.3) implies that $V(H^*) \cap C = V(C)$ and we obtain from this that the base group $\bar{H} = H^*/V(H^*)$ is a free product (in the variety $\mathfrak{U}$) of its subgroups $\bar{C} = C/V(C)$ and H (we recall that H is a free group in $\mathfrak{U}$). The subgroup $\bar{C}$ is the kernel of the epimorphism of F on the direct product $G \times H$; it is natural to call it the cartesian subgroup of the verbal wreath product $G_{\mathrm{wr}_{\mathfrak{U}}} H$. We have obtained the following lemma:

LEMMA 7.3: *The cartesian subgroup $\bar{C}$ of the verbal wreath product $F = G_{\mathrm{wr}_{\mathfrak{U}}} H$ is a $\mathfrak{U}$-free factor of the base group $\bar{H}$ and*

$$\bar{H} \simeq \bar{C}_{\mathfrak{U}} * H;$$

the group $\bar{C}$ is free in the variety $\mathfrak{U}$ and is generated freely by the set of elements

$$[g, h] \quad (1 \neq g \in G; 1 \neq h \in H).$$

COROLLARY: *Let $1 \neq g \in G$, $1 \neq h \in H$. Then the elements g and h cannot commute in the group F.*

Proof of the necessity of the conditions of Theorem B: We recall first that G and H must be residually nilpotent. It is worth remarking also that if f is a generalized periodic element, or a p-element, or an element of infinite p-height in one of the subgroups G or H, then it has the same properties in F.

We will consider now separately all the cases (i)–(iv).

(i) G and H contain no generalized periodic elements, i.e. they are residually torsion free nilpotent. We have already pointed out that this case follows from Šmelkin's Theorem [18].

(ii) Let g be an element of order p in G. If h is an element which has an infinite p-height in H, then $[g, h] = 1$ by Lemma 7.1 which contradicts Corollary of Lemma 7.3. Hence, $H \in \mathrm{res}\mathfrak{N}_p$.

We will now prove that $G \in \mathrm{res}\mathfrak{N}_p$. *Assume that there exists an element u of infinite p-height in G.* We show first that this assumption together with the residual nilpotence of F implies that H must be abelian. To prove this we can assume that H contains two non-unit elements $h_1 \neq h_2$ and we apply Lemma 7.2 to obtain two different elements $[g, h_1]$, $[u, h_2]$ from a free system of generators of $\bar{H}$; since by Lemma 7.1 the first element is a p-element and the second one has an infinite p-height, these elements must commute. This via the relative freeness of $\bar{H}$ implies that $\bar{H}$ is abelian, and so is H. Moreover, since H is relatively free it must be either free abelian or a free abelian group of finite exponent; clearly, F is isomorphic to a wreath product $G_{\mathrm{wr}}H$.

Consider first the case when H has a finite exponent. This exponent must be a power of p since we have already proved that in case (ii) H must belong to the case $\mathrm{res}\mathfrak{N}_p$. Take an arbitrary $1 \neq h \in H$. Once again we obtain that $[h, u] = 1$, which is impossible.

Now assume that H is free abelian. It is well known that the base group $\bar{H}$ can be considered as a free module over the integral group ring ZG: the module operation is defined by the conjugation by elements from G, the ZG-base of $\bar{H}$ is given by a system of free generators h_j $(j \in J)$ of H; we recall also that if $\varpi(ZG)$ is the augmentation ideal of ZG, then

$$\varpi^k(ZG) \cdot \bar{H} = \Big[\bar{H}, \underbrace{G, G, \ldots, G}_{k}\Big].$$

It is known (see, for instance [8]) that

$$0 \neq (g-1)(u-1) \in \bigcap_{k-1}^{\infty} \varpi^k(ZG).$$

We obtain from this that for every non-unit element $h \in H$,

$$1 \neq [[h, g], u] \in \bigcap_{k=1}^{\infty} \gamma_k(F),$$

which contradicts the assumption on the residual nilpotence of F. This completes the proof of (ii).

(iii) Let h be an element of order p in H. If now G contains an element g of infinite p-height, then $[g, h] = 1$, which contradicts Corollary of Lemma 7.2. Hence, $G \in \mathrm{res}\mathfrak{N}_p$.

Now assume that there exists in H two non-unit elements h_1 and h_2 of orders p and q respectively, $p \neq q$. If $1 \neq g \in G$, then $[g, h_1]$ is a p-element and it must commute with the q-element $[g, h_2]$. Once again, we conclude that $\bar{H}$, and hence H, is abelian; since H contains elements of finite order, it must have a finite exponent. This completes the proof of (iii).

(iv) We can assume that $r \geq 1$, $s \geq 1$. *Assume that for every $p_i(f)$ $(i \in I)$ there exists among the elements* (7.2) *an element of infinite p_i-height.* Consider the set of elements

(7.4) $$[g_\alpha, h_\beta] \quad (\alpha = 1, 2, \ldots, r; \beta = 1, 2, \ldots, s)$$

and let v be any right-normed commutator of elements (7.4); this commutator has an infinite p_i-height for each $p_i(f)$ $(i \in I)$.

Assume first that H is not nilpotent. This together with the relative freeness of H implies that $v \neq 1$. If now $f \in G$, then we pick an arbitrary element $h \neq h_\beta$ $(\beta = 1, 2, \ldots, s)$ and obtain via Lemma 7.2 that v must commute with the element $[f, h]$, i.e. a right-normed $(n+1)$th commutator formed by the elements (7.4) and the element $[f, h]$ is equal to 1; this contradicts the assumption on non-nilpotency of H. Similarly, if $f \in H$ we take $g \neq g_\alpha$ $(\alpha = 1, 2, \ldots, r)$ and obtain that v must commute with the element $[f, g]$, which is impossible.

Consider now the case when H is nilpotent (and torsion free). The generalized periodic element in this case must belong to G. Furthermore, for every prime p the elements of the relatively free torsion free nilpotent group cannot have an infinite p-height. Hence we can rewrite the assumption made in the beginning of the proof as:

For every $p_i(f)$ $(i \in I)$ there exists among the elements

$$g_\alpha \quad (\alpha = 1, 2, \ldots, r)$$

an element of infinite p_i-height.

Let x be an arbitrary element which has an infinite order modulo the commutator subgroup H'. The commutator $[x, g_1, g_2, \ldots, g_r]$ has an infinite p-height for every $i \in I$. Hence

$$[x, g_1, g_2, \ldots, g_r, f] = 1. \tag{7.5}$$

We prove now that the relation (7.5) in F is impossible under the assumption that H is torsion free nilpotent. Indeed, let $\sqrt{H'} = \{h \in H |\ h^{n(h)} \in H'\}$. We have the epimorphism $G \mathrm{wr}_{\mathfrak{U}} H \longrightarrow G \mathrm{wr} H_0$, defined by the epimorphism $H \longrightarrow H_0$. The relation (7.5) yields in $G \mathrm{wr} H_0$, for the element $\bar{x}$, the image of x in H:

$$[\bar{x}, g_1, g_2, \ldots, g_r, f] = 1. \tag{7.6}$$

The group H_0 is free abelian and we consider now, as in (ii), the base group $\bar{H}_0$ of the wreath product $G \mathrm{wr} H_0$ as a free ZG-module; the relation (7.6) now yields

$$(g_1 - 1)(g_2 - 2) \cdots (g_r - 1) \cdot \bar{x} = 0. \tag{7.7}$$

It is well known that the element $(g_1 - 1)(g_2 - 1) \cdots (g_r - 1) \in ZG$ cannot be a zero divisor if G is torsion free; since $\bar{x}$ is an element from a free ZG-module

$\bar{H}$, we see that the relation (7.7) is impossible and the proof of (iv) is completed. Theorem B is proven. ∎

Proof of the necessity of the conditions of Theorem C: We recall once again that the condition $\bigcap_{k=1}^{\infty} \varpi^k(ZF) = 0$ implies that $\bigcap_{n=1}^{\infty} \gamma_k(F) = 1$. We obtain from this that one of the conditions (i)–(iv) of Theorem B must hold; we only need to eliminate the second subcase in (iii). But if H is abelian of finite exponent $n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$ and $k \geq 2$, then it is known that $\bigcap_{k=1}^{\infty} \varpi^k(ZH) \neq 0$, i.e. this case cannot occur and the proof is completed.

8. Concluding remarks

We have already pointed out that Theorem A should be compared with Šmelkin's Theorem [18] and that our methods are completely different from the method in [18]; one of the reasons for this is that there is no analog of the Campbell–Hausdorff formulae for restricted Lie algebras. There is, however, one more essential difference between the case of torsion free nilpotent groups and the modular case which we study. If H and G are residually torsion free nilpotent, then the Lie algebra $L(F)$ is a verbal wreath product of the relatively free Lie algebra $L(H)$ and the Lie algebra $L(G)$; this is not true for the algebras $L_p(G)$ which we study, and in fact little is known about the algebras $L_p(F)$.

We have already mentioned the conjecture about the necessary conditions for the residual nilpotence of the groups of the type $\bar{S} = S/V(N)$; the author has some results in this direction. We formulate without proof one of these results.

Let S be a free group, $N \lhd S$, $G = S/N$ and $\bar{S} = S/V(N)$. The augmentation ideal $\varpi(Z\bar{S})$ is residually nilpotent iff one of the three conditions of Theorem C holds.

The sufficiency of these conditions follows immediately from Theorem C and Šmelkin's imbedding theorem.

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